

# Deformation Quantization for actions of the affine group (PRELIMINARY VERSION)

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## Abstract

We define a universal deformation formula (UDF) for the actions of the affine group on Fréchet algebras. More precisely, starting with any associative Fréchet algebra  $\mathbb{A}$  which the affine group  $\mathbb{S} \simeq ax+b$  acts on in a strongly continuous and isometrical manner, the UDF produces a family of topological associative algebra structures on the space  $\mathbb{A}^\infty$  of smooth vectors of the action deforming the initial product. The deformation field obtained is based over an infinite dimensional parameter space naturally associated with the space of pseudo-differential operators on the real line. This note also presents some geometrical aspects of the UDF and in particular its relation with hyperbolic geometry.

## 1 Admissible functions on symmetric spaces

### 1.1 Von Neumann's formula in the flat case

In Weyl's quantization of the flat plane  $\mathbb{R}^2$ , the formula for the composition of Weyl's symbols can be expressed in terms of an oscillatory integral three-point kernel whose phase is proportional to the area of a Euclidean triangle. More precisely, Weyl's product of two Schwartz symbols  $a, b \in \mathcal{S}(\mathbb{R}^2)$  is given by the following expression, here involving the (real) deformation parameter  $\theta$ :

$$(a \star_\theta^w b)(x) = \frac{1}{\theta^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{\frac{i}{\theta} S(x,y,z)} a(y) b(z) dy dz . \quad (1)$$

In this formula, originally due to Von Neumann [vN31],  $S(x, y, z)$  is a constant multiple of the symplectic area of the Euclidean triangle with vertices  $x, y$  and  $z$  in  $\mathbb{R}^2$ .

Only two geometric properties of  $S$  are actually sufficient for proving associativity of Weyl's product directly at the level of the above formula—i.e. without reference to any operator representation of symbols [Bi02]. The first geometric property is the fact that given any three points  $x, y$  and  $z$  in the plane, one has:

$$S(x, s_x(y), z) = -S(x, y, z) , \quad (2)$$

where  $s_x$  denotes the geodesic symmetry of the plane centred at point  $x$ . The second geometric property is the additivity of triangle areas: for all  $x, y, z$  and  $m$  in the plane, one has:

$$S(x, y, m) + S(y, z, m) + S(z, x, m) = S(x, y, z). \quad (3)$$

These properties naturally lead to the following definitions in the more general case of symplectic symmetric spaces.

### 1.2 Symplectic symmetric spaces

For convenience, we recall in this subsection the notion of symplectic symmetric space ( see [Bi95, BCG95] for details).

**Definition 1.1** A **symplectic symmetric space** is a triple  $(M, \omega, s)$  where  $(M, \omega)$  is a connected smooth manifold,  $\omega$  is a non-degenerate two-form on  $M$  and  $s : M \times M \rightarrow M$  is a smooth map such that

- (i) For any  $x$  in  $M$ ,  $s_x : M \rightarrow M : y \mapsto s(x, y)$  is a  $\omega$ -preserving diffeomorphism of  $M$ , which is involutive ( $s_x^2 = \text{id}_M$ ) and which admits  $x$  as isolated fixed point. The map  $s_x$  is called the **symmetry** at  $x$ .

- (ii) For any  $x, y$  in  $M$  one has :

$$s_x s_y s_x = s_{s_y x} .$$

The data of the map  $s$  then uniquely determines an affine connection  $\nabla$  on  $M$  which is invariant under the symmetries. The connection turns out to be torsion free and with respect to which the two-form  $\omega$  is parallel. In particular,  $\omega$  is symplectic on  $M$ .

**Definition 1.2** A triple  $t = (\mathfrak{g}, \sigma, \Omega)$ , where  $(\mathfrak{g}, \sigma)$  is a **involutive Lie algebra**—i.e.  $\mathfrak{g}$  is a finite dimensional real Lie algebra and  $\sigma$  is an involutive automorphism of  $\mathfrak{g}$ —and where  $\Omega$  is an element of  $\bigwedge^2 \mathfrak{g}$ , is called a **symplectic triple** if the following properties are satisfied:

- (i) Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  where  $\mathfrak{k}$  (resp.  $\mathfrak{p}$ ) is the  $+1$  (resp.  $-1$ ) eigenspace of  $\sigma$ , then  $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$  and the representation of  $\mathfrak{k}$  on  $\mathfrak{p}$ , given by the adjoint action, is faithful.
- (ii)  $\Omega$  is a Chevalley 2-cocycle for the trivial representation of  $\mathfrak{g}$  on  $\mathbb{R}$  such that for any  $X$  in  $\mathfrak{k}$ ,  $i(X)\Omega = 0$  and such that the restriction of  $\Omega$  to  $\mathfrak{p} \times \mathfrak{p}$  defines a symplectic structure.

**Proposition 1.1** There is a bijection between the set of isomorphism classes of simply connected symplectic symmetric spaces and the set of isomorphism classes of symplectic triples.

Let us briefly recall how this correspondence is made. If  $(M, \omega, s)$  is a symplectic symmetric space, the group  $G$  generated by products of an even number of symmetries is a transitive Lie transformation group of  $M$ .  $G$  is called the **transvection group** of the symplectic symmetric space. We associate to  $(M, \omega, s)$  a symplectic triple  $t = (\mathfrak{g}, \sigma, \Omega)$  as follows. The Lie algebra  $\mathcal{G}$  is the Lie algebra of the transvection group  $G$ . Let us choose a base point  $o$  in  $M$  and let  $\tilde{\sigma}$  be the involutive automorphism of  $G$  obtained by “conjugation” by  $s_o$ . Then  $\sigma$  is the differential of  $\tilde{\sigma}$  at the neutral element  $e$  of  $G$ . If  $\pi : G \rightarrow M : g \mapsto g.o$  denotes the projection associated to the choice of base point  $o$ , then  $\Omega$  is the 2-form on  $\mathfrak{g}$  (identified with  $G_e$ ),  $\Omega = \pi^* \omega_o$ . The subalgebra  $\mathfrak{k}$  of  $\mathfrak{g}$  is known to be isomorphic to the holonomy algebra for the Loos connection  $\nabla$  on the symmetric space  $M = G/K$ —where  $K$  is the isotropy at  $o$ ; its Lie algebra is isomorphic to  $\mathfrak{k}$ . In this context any geodesic through point  $o$  is of the form  $t \mapsto \exp(tX)$  where  $X \in \mathfrak{p}$ .

In view of UDF’s, we will be concerned with symmetric spaces underlying Lie group manifolds [BCSV07].

**Definition 1.3** A (symplectic) symmetric space, or more generally a homogeneous space,  $M$  of dimension  $m$  is **locally of group type** if there exists a  $m$ -dimensional (symplectic) Lie subgroup  $\mathbb{S}$  of its automorphism group which acts freely on one of its orbits in  $M$ . One says that it is **globally of group type** if it is locally and if  $\mathbb{S}$  has only one orbit.

In the global case, for every choice of a base point  $o$  in  $M$ , the map  $\mathbb{S} \rightarrow M : g \mapsto g.o$  is a  $\mathbb{S}$ -equivariant diffeomorphism.

**Example 1.1** In the present note, we’ll be mainly concerned with symmetric surfaces (i.e.  $\dim M = 2$ ) which the affine group  $\mathbb{S} = ax + b$  acts on by automorphisms. The only globally  $\mathbb{S}$ -type symmetric surfaces are (see e.g. [Bi98]) the hyperbolic plane  $\mathbb{D} := SL(2, \mathbb{R})/SO(2)$  and the co-adjoint orbit of the Poincaré group  $\mathbb{M} := SO(1, 1) \times \mathbb{R}^2/\mathbb{R}$ . In particular, one has the following  $\mathbb{S}$ -equivariant symplectic identifications:  $\mathbb{M} = \mathbb{S} = \mathbb{D}$  and one may therefore think to the space  $\mathbb{M}$  as a ‘curvature contraction’ of the hyperbolic plane  $\mathbb{D}$ .

We now detail the above example and denote by  $\mathfrak{s}$  the (solvable) non-Abelian two-dimensional real Lie algebra which we present as generated by the elements  $H$  and  $E$  with relations  $[H, E] = 2E$ . Setting  $\mathfrak{a} := \mathbb{R} H$  and  $\mathfrak{n} := \mathbb{R} E$  realizes  $\mathfrak{s}$  as the semi-direct product  $\mathfrak{s} = \mathfrak{a} \ltimes \mathfrak{n}$ . The corresponding connected simply connected Lie group  $\mathbb{S}$  is of the solvable exponential type and the map

$$\mathfrak{s} \rightarrow \mathbb{S} : (a, \ell) := aH + \ell E \mapsto \exp(aH) \cdot \exp(\ell E) \quad (4)$$

is a global diffeomorphism. Within these notations, the group law reads as

$$(a, \ell).(a', \ell') = (a + a', e^{-2a'}\ell + \ell') .$$

**Lemma 1.1** *Consider the solvable symmetric surface  $\mathbb{M} = SO(1, 1) \times \mathbb{R}^2/\mathbb{R}$ . Then,*

- (i) *under the identification  $\mathbb{M} = \mathbb{S}$ , the coordinate system (4) is Darboux ( $\omega$  is proportional to  $da \wedge d\ell$ ) and the symmetry map reads:*

$$s_{(a, \ell)}(a', \ell') = (2a - a', 2 \cosh(2(a - a'))\ell - \ell') .$$

- (iii) *The holonomy group  $K$  at  $o := (0, 0)$  is isomorphic to  $\mathbb{R}$  and its action reads:*

$$\kappa.(a, \ell) = (a, \ell - \kappa \sinh(a)) .$$

### 1.3 Admissibility

To a symmetric space  $M$ , one may attach a natural ‘group-like’ cohomological complex on multiple-point functions.

**Definition 1.4** *Let us define the  $k$ -th co-chain space  $\mathbb{CP}^k(M)$  as the space of all complex valued smooth functions on  $M^k$  that are invariant under the (diagonal) action of the symmetries on  $M^k$ . Then, the formula*

$$\delta F(x_0, \dots, x_k) := \sum_i (-1)^i F(x_0, \dots, \hat{x}_i, \dots, x_k)$$

*defines a cohomology operator  $\delta : \mathbb{CP}^k(M) \rightarrow \mathbb{CP}^{k+1}(M)$ .*

Observe that, in this framework, property (3) simply amounts to cocyclicity of  $S \in \mathbb{CP}^3(\mathbb{R}^2)$ . Regarding property (2) we observe that, given a geodesic  $\gamma$  and given a point  $y$  in a symmetric space, the curve traced out by  $s_x s_z(y)$  where  $x$  and  $z$  run in  $\gamma$  is in general not a geodesic. It is rather the orbit of a one-parameter transvection subgroup  $\{A(t)\}_{t \in \mathbb{R}}$ , antifixed under the conjugation by  $s_z$ , and realizing the geodesic  $\gamma$ :

$$A(t).z = \gamma(t) .$$

We then make the following definition, slightly stronger than (2).

**Definition 1.5** *Let  $M$  be a geodesically convex symmetric space. A three-cochain  $S \in \mathbb{CP}^3(M)$  is called **admissible** if additionally to (2) one has, for all  $x, y, z \in M$ , that:*

$$S(x, A(t).y, z) = S(x, y, z) . \tag{5}$$

The following fact stresses the relevance of admissible functions on symmetric spaces (for details, see [Bi02]).

**Proposition 1.2** *Consider a geodesically convex symplectic symmetric space  $M$  and assume  $S \in \mathbb{CP}^3(M)$  is admissible and cocyclic. Then the product defined by the following formula:*

$$(a \star_\theta b)(x) = \frac{1}{\theta^2} \int_{M \times M} e^{\frac{i}{\theta} S(x, y, z)} a(y) b(z) dy dz \quad (a, b \in C_c^\infty(M))$$

*is formally associative.*

However, when curvature is present, admissibility and cocyclicity are, in the non-degenerate case (i.e.  $S$  of Morse type), incompatible conditions. Hence, following a standard paradigm in theoretical physics, the idea is to start from an admissible non-cocyclic function and then define a deformation framework for it where associativity holds.

As a starting point in this program, we now give a convenient description of admissible functions on strictly geodesically convex symmetric surfaces in terms of admissible functions on the flat plane.

**Example 1.2** Let  $(M, \omega, s)$  be a simply connected strictly geodesically convex symplectic symmetric space of dimension two. It can then be realized as a coadjoint orbit  $\mathcal{O}$  of a three dimensional Lie group  $G$  in the dual  $\mathfrak{g}^*$  of its Lie algebra  $\mathfrak{g}$ . Denoting by  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  the decomposition into  $(\pm 1)$ -eigenspaces of  $\sigma$ , the inclusion  $\mathfrak{p} \rightarrow \mathfrak{g}$  induces a canonical projection  $\Pi : \mathfrak{g}^* \rightarrow \mathfrak{p}^*$  whose restriction to  $\mathcal{O}$  defines, in this case, a global diffeomorphism

$$\Pi : \mathcal{O} \xrightarrow{\sim} \mathfrak{p}^* .$$

Denote by  $o$  the point of  $\mathcal{O}$  corresponding to the origin  $0$  of  $\mathfrak{p}^*$  under the diffeomorphism  $\Pi$ , and denote by  $K$  the stabilizer of  $o$  in  $G$ . For every  $x \in \mathcal{O}$  consider the associated (globally well-defined) mid-point map  $x \rightarrow \frac{x}{2}$  defined by the following property:

$$s_{\frac{x}{2}}(o) := x .$$

**Proposition 1.3** *View  $(\mathfrak{p}^*, \Omega)$  as the flat symplectic plane and consider a  $K$ -invariant admissible function  $S^0$  on  $\mathfrak{p}^*$  (with respect to the flat structure). Then, the formula:*

$$S(x, y, z) := S^0(0, \Pi(s_{\frac{x}{2}}(y)), \Pi(s_{\frac{x}{2}}(z))) \quad (6)$$

*defines an admissible function on  $M = \mathcal{O}$ . Moreover, every admissible function on  $M$  is obtained this way.*

*Proof.* We first observe that the diffeomorphism  $\Pi$  establishes a bijection between the  $\exp(tX)$ -orbits ( $X \in \mathfrak{p}$ ) in  $\mathcal{O}$  and the straight lines in  $\mathfrak{p}^*$ . Indeed, for  $x \in \mathcal{O}$  and  $X \in \mathfrak{p}$ , one has  $\langle \text{Ad}^*(\exp(tX))x - x, X \rangle = 0$ . Which means that the  $x$ -translated  $\exp(tX)$ -orbit of  $x$  lies in the plane in  $\mathfrak{g}^*$  orthogonal to  $X \in \mathfrak{p}$ . This plane is generated by the kernel  $\mathfrak{k}^*$  of the projection  $\Pi : \mathfrak{g}^* \rightarrow \mathfrak{p}^*$  and an element  $X^\perp$  of  $\mathfrak{p}^*$  orthogonal to  $X$ . In particular, it projects onto the line directed by  $X^\perp$ .

Now consider the two-point function  $u(x, y) := S(o, x, y)$  on  $M$  induced by the data of an admissible function  $S$  on  $M$ . This function corresponds to a two-point function  $u^0$  on  $\mathfrak{p}^*$  via the diffeomorphism  $\Pi$ . By admissibility (cf. (5)) and the above observation, one has  $u^0(\xi, \eta) = u^0(\xi, \eta + t\xi)$  for all  $t \in \mathbb{R}$ . Which is precisely the property of admissibility for a two-point function with respect to the flat structure on  $\mathfrak{p}^*$ . The rest then follows from Proposition 3.3 in [Bi02]. ■

**Definition 1.6** *By virtue of the above proposition, the Euclidean triangle symplectic area on  $\mathfrak{p}^*$  defines an admissible three-point function on  $M$ . The latter will be denoted  $S_{\text{can}}$  and called the canonical admissible function.*

Observe furthermore that any odd function of a multiple of  $S_{\text{can}}$  defines an admissible three-point function on  $M$ . These particular admissible functions correspond to the ones which in addition to their  $\mathfrak{k}$ -symmetry enjoy a full  $\mathfrak{sp}(1, \mathbb{R})$ -symmetry.

**Example 1.3** *Within the coordinate system (4), the canonical admissible function  $S_{\text{can}}$  on  $\mathbb{M} = SO(1, 1) \times \mathbb{R}^2/\mathbb{R}$  has the following expression:*

$$S_{\text{can}}(x_0, x_1, x_2) = \frac{1}{2} \sum_{0,1,2} \sinh(2(a_0 - a_1)) \ell_2 , \quad (7)$$

where  $x_i = (a_i, \ell_i)$  ( $i = 0, 1, 2$ ).

## 2 Strict quantizations of $SO(1, 1) \times \mathbb{R}^2/\mathbb{R}$

In this section, we describe all the invariant deformation quantizations on the symplectic symmetric surface  $\mathbb{M} := SO(1, 1) \times \mathbb{R}^2/\mathbb{R}$ . We begin by slightly generalizing the WKB-quantization constructed in [Bi02]. What follows also provides a rigorous framework for statements made in [BDRS04].

Endowing  $\mathbb{S}$  with any left-invariant Haar measure, the Darboux map (4) yields the identifications

$$L^2(\mathbb{S}) = L^2(\mathfrak{s}) = L^2(\mathbb{M}) .$$

The partial Fourier transform with respect to the variable  $\ell$  will be denoted by

$$\begin{aligned}\mathcal{F}_N : L^2(\mathbb{S}) &\rightarrow L^2(\tilde{\mathbb{S}}) \\ \mathcal{F}_N(u)(a, \alpha) &:= \int_{\mathbb{N}} e^{-i\alpha\ell} u(a, \ell) \, d\ell ,\end{aligned}$$

where  $\tilde{\mathbb{S}} := \{(a, \alpha)\}$  denotes the space where Fourier transformed functions are defined on.

**Definition 2.1** *The twisting map is the one-parameter family of diffeomorphisms of  $\tilde{\mathbb{S}}$  defined by*

$$\varphi_\theta(a, \alpha) := \left( a , \frac{1}{2\theta} \sinh(2\theta\alpha) \right) \quad \theta \in \mathbb{R}.$$

Observe that, denoting by  $\tilde{\mathcal{S}}$  the Schwartz function space on  $\tilde{\mathbb{S}}$ , on has

$$\varphi_\theta^*(\tilde{\mathcal{S}}) \subset \tilde{\mathcal{S}}.$$

Therefore, for every invertible operator multiplier  $\Theta \in \mathcal{O}_M(\mathbb{R})$ , the following linear map defined on the Schwartz space  $\mathcal{S}$  on  $\mathbb{S}$  (with respect to coordinates  $(a, \ell)$ ) takes its values in the tempered smooth functions on  $\mathbb{S}$ :

$$U_\theta^\Theta := \mathcal{F}_N^{-1} \circ \mathcal{M}_\Theta \circ (\varphi_\theta^{-1})^* \circ \mathcal{F}_N : \mathcal{S} \rightarrow \mathcal{S}' . \quad (8)$$

Its range will be denoted

$$\mathcal{E}_\theta^\Theta := U_\theta^\Theta(\mathcal{S}) \subset \mathcal{S}' ;$$

and we set

$$(U_\theta^\Theta)^{-1} : \mathcal{E}_\theta^\Theta \xrightarrow{\sim} \mathcal{S}$$

for the associated (inverse) linear isomorphism. The space  $\mathcal{E}_\theta^\Theta$  carries the transported Schwartz topology. Note that following the same argument as in [Bi02], one has the inclusion:

$$\mathcal{E}_\theta^\Theta \supset \mathcal{S} .$$

**Theorem 2.1** *Let  $\Theta$  be any invertible element of  $\mathcal{O}_M(\mathbb{R})$  and consider  $\theta > 0$ .*

(i) *Let  $u$  and  $v$  in  $C_c^\infty(\mathbb{M})$  and  $x_0 \in \mathbb{M}$ . Then, the formula<sup>1</sup>:*

$$(u \star_\theta^\Theta v)(x_0) := \frac{1}{2\pi\theta^2} \int_{\mathbb{M} \times \mathbb{M}} \cosh(2(a_1 - a_2)) \frac{\Xi(a_2 - a_0)\Xi(a_0 - a_1)}{\Xi(a_2 - a_1)} e^{\frac{i}{\theta} S_{\text{can}}(x_0, x_1, x_2)} u(x_1) v(x_2) \, dx_1 \, dx_2 ,$$

where

$$\Xi(\theta t) := \frac{1}{(\varphi_\theta^* \Theta)(t)} ,$$

extends to  $\mathcal{E}_\theta^\Theta$  as an associative product. The pair  $(\mathcal{E}_\theta^\Theta, \star_\theta^\Theta)$  is then a Fréchet algebra.

(ii) *The formula*

$$\text{Tr}_\theta^\Theta(u) = \frac{1}{\Theta(0)} \int_{\mathbb{M}} u \quad (u \in \mathcal{S})$$

extends to  $\mathcal{E}_\theta^\Theta$  as a trace  $\text{Tr}_\theta^\Theta : \mathcal{E}_\theta^\Theta \rightarrow \mathbb{C}$  for  $\star_\theta^\Theta$ .

(iii) *The algebra  $(\mathcal{E}_\theta^\Theta, \star_\theta^\Theta)$  is strongly closed if and only if*

$$\Xi(t)\Xi^\vee(-t) = \cosh(2t) . \quad (9)$$

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<sup>1</sup>Formula (i) below has been announced in [BDRS04].

*Proof.* Item (i) is obtained by a long but straightforward computation, entirely similar to the one in [Bi02] (see the proof of Theorem 6.13 page 311), where one transports Weyl's product on  $\mathcal{S} = \mathcal{S}_{(a,\ell)}$  to  $\mathcal{E}_\theta^\Theta$  via  $U_\theta^\Theta$ . The trace formula is obtained by combining  $\text{Tr}^\Theta = \text{Tr}^w \circ (U_\theta^\Theta)^{-1}$  where  $\text{Tr}^w(u) := \int_{\mathbb{M}} u$  with the fact that  $\int \circ \mathcal{F}_N^{-1} = \delta_0$ . At last, using strong closedness of Weyl's product, one gets

$$\text{Tr}^\Theta(u \star_\theta^\Theta v) = \int (U_\theta^\Theta)^{-1}(u) (U_\theta^\Theta)^{-1}(v).$$

Which equals  $\langle (U_\theta^\Theta)^{-1}(u), \overline{(U_\theta^\Theta)^{-1}(v)} \rangle_{L^2(\mathbb{M})}$ . Setting  $f_\theta(t) := f(\theta t)$  and  $f^\vee(t) := f(-t)$ , the latter becomes

$$\begin{aligned} \langle \mathcal{F}_N^{-1}(\Theta_\theta \varphi_\theta^\star \mathcal{F}_N u), \mathcal{F}_N^{-1} \left[ \overline{(\Theta_\theta \varphi_\theta^\star \mathcal{F}_N u)} \right]^\vee \rangle_{L^2(\mathbb{M})} &= \int \Theta_\theta \Theta_\theta^\vee \varphi_\theta^\star (\mathcal{F}_N(u) [\mathcal{F}_N(v)]^\vee) = \\ &= \int (\varphi_\theta^{-1})^\star (\Theta_\theta \Theta_\theta^\vee) \cdot |\text{Jac}_{\varphi_\theta^{-1}}| \cdot \mathcal{F}_N(u) [\mathcal{F}_N(v)]^\vee. \end{aligned}$$

The last member equals  $\int uv$  if and only if  $(\varphi_\theta^{-1})^\star (\Theta_\theta \Theta_\theta^\vee) \cdot |\text{Jac}_{\varphi_\theta^{-1}}| = 1$ , which is the announced condition. ■

**Remark 2.1** (i) The invertible element  $\Theta$  can be considered as a parameter in the construction. Moreover, it can itself depend on the real parameter  $\theta$  as well.

(ii) By defining  $\langle a, b \rangle_\Theta := \text{Tr}^\Theta(a \star^\Theta \bar{b})$ , one could study the associated field of Hilbert  $G(\mathbb{M})$ -algebras in the same line as in [Bi02]. We will not follow this route here, but rather focus on the square integrable case associated with the unitary condition (9).

The manifold  $\mathbb{M} \times \mathbb{M} \times \mathbb{M}$  admits a distinguished transformation, namely one has

**Lemma 2.1** [Qi97, Bi02]

(i) Given any triple of points  $x, y$  and  $z$  in  $\mathbb{M}$ , the equation

$$s_x s_y s_z(t) = t$$

admits a unique solution  $t \in \mathbb{M}$ .

(ii) The associated map

$$\Phi : \mathbb{M} \times \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{M} \times \mathbb{M} \times \mathbb{M} : (x, y, z) \mapsto (t, s_x(t), s_y s_z(t))$$

is a global diffeomorphism.

**Lemma 2.2**

$$\text{Jac}_\Phi(x_0, x_1, x_2) = 16 \cosh(2(a_0 - a_1)) \cosh(2(a_1 - a_2)) \cosh(2(a_2 - a_0)).$$

*Proof.* This is a straightforward computation based on the following formula for the mid-point map:

$$m : M \times M \rightarrow M : (x, y) \mapsto m(x, y) = \left( \frac{1}{2}(a_x + a_y), \frac{1}{2}(\ell_x + \ell_y) \text{sech}(2(a_x - a_y)) \right),$$

defined by the relation

$$s_{m(x,y)} x = y.$$

One has

$$\Phi^{-1}(x, y, z) = (m(x, y), m(y, z), m(z, x));$$

and a computation yields

$$\text{Jac}_{\Phi^{-1}}(x_0, x_1, x_2) = \frac{1}{16} \text{sech}(2(a_0 - a_1)) \text{sech}(2(a_1 - a_2)) \text{sech}(2(a_2 - a_0)).$$

One then obtains the announced formula by using the relation:  $\text{Jac}_\Phi = (\Phi^\star \text{Jac}_{\Phi^{-1}})^{-1}$ . ■

The latter together with the unitary condition (9) yield

**Corollary 2.1** *Let  $\theta > 0$ . Then,*

(i) *the following formula*

$$u \star_\theta v(x) := \frac{1}{\theta^2} \int_{\mathbb{M} \times \mathbb{M}} [Jac_\Phi(x, y, z)]^{\frac{1}{2}} e^{\frac{i}{\theta} S_{\text{can}}(x, y, z)} u(y) v(z) \, dy \, dz \quad (u, v \in C_c^\infty(\mathbb{M})) \quad (10)$$

*extends to the space  $L^2(\mathbb{M})$  as an associative product.*

(ii) *The algebra  $(L^2(\mathbb{M}), \star_\theta)$  becomes a Hilbert algebra when one endows  $L^2(\mathbb{M})$  with its natural Hilbert space structure.*

(iii) *The transvection group  $G = G(\mathbb{M})$  acts on the above algebra by unitary automorphisms.*

remains to give a geometrical meaning to the co-boundary factor appearing in the prproduct formula. For this, we observe that the affine manifold  $\mathbb{M}$  admits a canonical invariant foliation with respect to which  $\ell$ -independent functions on  $\mathbb{M}$  correspond to leafwise constant functions.

**Lemma 2.3** [Bi02] *The symmetric surface  $\mathbb{M}$  admits a unique one-dimensional distribution  $\mathcal{L} \subset T(\mathbb{M})$  which is invariant under the transvection group. The corresponding (Lagrangian) foliation of  $\mathbb{M}$  is a fibration by geodesics.*

The twisting map now appears as a one parameter family of transformations of  $\mathbb{M}/\mathcal{L}$ . Moreover, one notes

**Corollary 2.2** *The factor  $\frac{\Xi(a_2 - a_0)\Xi(a_0 - a_1)}{\Xi(a_2 - a_1)}$  corresponds to  $\exp(\delta\Xi)$  where  $\Xi$  is a symmetry invariant  $\mathcal{L} \times \mathcal{L}$ -leafwise constant 2-cochain in  $\text{CP}^2(\mathbb{M})$ .*

We end this section with a remark concerning invariant formal star products on  $\mathbb{M}$ . First we observe that following the same lines as in the proof of Proposition 4.2. in [BBM07], one obtains

**Proposition 2.1** *Consider a  $\mathcal{O}_M(\mathbb{R})$ -valued smooth function  $\Theta : ]-\epsilon, \epsilon[ \rightarrow \mathcal{O}_M(\mathbb{R}) : \theta \mapsto \Theta_\theta$  such that  $\Theta_\theta$  is invertible for all  $\theta$  and such that  $\Theta_0 \equiv 1$ . Let  $u$  and  $v$  be smooth and compactly supported and consider the associated product*

$$u \star_\theta^\Theta v = \frac{1}{\theta^2} \int_{\mathbb{M} \times \mathbb{M}} |Jac_\Phi|^{\frac{1}{2}} \exp(\delta\Xi_\theta) e^{\frac{i}{\theta} S_{\text{can}}} u \otimes v ,$$

*as in Theorem 2.1 (i). Then the function  $\theta \mapsto u \star_\theta^\Theta v$  is a smooth  $C^\infty(\mathbb{M})$ -valued function whose Taylor series at  $\theta = 0$  defines a symmetry invariant formal star product  $\tilde{\star}_\theta^\Theta$  on  $C^\infty(\mathbb{M})[[\theta]]$ .*

The remark is then that every invariant star product on  $\mathbb{M}$  can be seen as an asymptotic expansion of an oscillatory integral of the above type. Indeed, through the identification  $\mathbb{S} = M$ , the formal star product  $\tilde{\star}_\theta^1$  can be viewed as a left-invariant formal star product on the symplectic Lie group  $\mathbb{S}$  with an additional  $K$ -invariance. Every other  $G$ -invariant star product on  $(\mathbb{M}, \omega)$  in the same  $G$ -characteristic class can therefore be obtained by intertwining  $\tilde{\star}_\theta^1$  by a left-invariant formal equivalence (i.e. an element of  $\mathcal{U}(\mathfrak{s})[[\theta]]$ ) which commutes with the  $K$ -action. Within the coordinate system (4), the left-invariant vector fields on  $\mathbb{S}$  corresponding to elements  $H$  and  $E$  of  $\mathfrak{s}$  have the following expressions:

$$\tilde{H} = \partial_a - 2\ell\partial_\ell , \text{ and } \tilde{E} = \partial_\ell .$$

While the fundamental vector field

$$Z_c^* := \sinh(a)\partial_\ell$$

is a generator of the  $K$ -action. From this, one sees that each term of the above mentioned equivalence must be polynomial in  $\tilde{E}$  only. In other words, the conjugation by  $\mathcal{F}_N$  of the equivalence is a multiplication by an (invertible) formal function. Combining the above observation with Formula (8) basically yields the following proposition

---

<sup>2</sup> $L^2(\mathbb{M})$  denotes the space of square-integrable functions with respect to the Liouville measure.

**Proposition 2.2** *Let  $\Theta \in \mathbb{C}[t][[\theta]]$  with non-zero constant leading term. Then, the formal star product defined as*

$$\tilde{\star}_\theta^\Theta := \mathcal{F}_N^{-1} \circ \mathcal{M}_\Theta \circ \mathcal{F}_N(\tilde{\star}_\theta^1)$$

*is  $G$ -invariant (in the above formula  $\mathcal{M}_\Theta$  denotes the multiplication operator:  $\mathcal{M}_\Theta(f) := \Theta f$ ). Moreover, every  $G$ -invariant formal star product on  $(\mathbb{M}, \omega)$  is of this form.*

*Proof.* It remains to prove the second assertion, which follows from the fact that changing  $G$ -characteristic class amounts to change the parameter. Indeed, one knows that the  $G$ -equivalence classes of invariant star products on  $(\mathbb{M}, \omega)$  are canonically parametrized by the formal series with coefficients in the second  $G$ -equivariant de Rham cohomology space  $H_G^2(\mathbb{M})$  [BBG98]. The latter is isomorphic to  $\mathbb{C}$  as generated by  $[\omega]$ . Hence every  $G$ -invariant class is of the form  $\Theta([\omega])$  where  $\Theta = \Theta(\theta)$  is a formal function. One then concludes by Proposition 4.3 in [BB03]. ■

### 3 Universal deformation Formulae

#### 3.1 Oscillatory integrals

We let  $E$  be a complex Fréchet space with topology defining family of seminorms  $\{|\cdot|_j\}_{j \in \mathbb{N}}$ . Let  $G$  be a solvable exponential Lie group with Lie algebra  $\mathfrak{g}$  and consider a function  $\mathbf{m} \in C^\infty(G, \mathbb{C})$  which is nowhere vanishing. We then define the following function space:

$$\mathcal{B}_E^{\mathbf{m}}(G) := \{F \in C^\infty(G, E) \text{ such that } \forall P \in \mathcal{U}(\mathfrak{g}) ; j \in \mathbb{N} \text{ there exists } C > 0 \text{ such that } |\tilde{P}.F|_j < C |\mathbf{m}| \} ;$$

where  $\tilde{P}$  denotes the left-invariant differential operator on  $G$  associated with the element  $P$  of the universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$ .

**Definition 3.1** *An everywhere non-zero function  $\mathbf{m} \in C^\infty(G, \mathbb{C})$  is called a **weight** if  $\mathbf{m} \in \mathcal{B}_\mathbb{C}^{\mathbf{m}}$ .*

Let  $C_b(G, E)$  be the space of  $E$ -valued bounded continuous functions on  $G$ . The group  $G$  then acts on the latter space via the right regular representation. Consider the subspace  $C_u(G, E)$  of  $C_b(G, E)$  constituted by the uniformly continuous functions. The following lemma is essentially standard.

**Lemma 3.1** *Let  $\mathbf{m}$  and  $\mathbf{m}'$  be weight functions. Then,*

- (i) *the group  $G$  acts on  $C_u(G, E)$  isometrically and strongly continuously. The space  $[C_u(G, E)]_\infty$  of smooth vectors for this action coincides with  $\mathcal{B}_E^1(G)$ .*
- (ii) *On  $\mathcal{B}_E^1(G)$ , the following seminorms:*

$$|a|_{P,j} := \sup_G \{ |\tilde{P}.a|_j \} \quad (P \in \mathcal{U}(\mathfrak{g})),$$

*induce the natural Fréchet topology on  $[C_u(G, E)]_\infty$  (cf. [Wa72]).*

*Analogously, on  $\mathcal{B}_E^{\mathbf{m}}(G)$  the seminorms*

$$|a|_{P,j} := \sup_G \left\{ \left| \frac{1}{\mathbf{m}} \tilde{P}.a \right|_j \right\}$$

*define a Fréchet topology.*

- (iii) *The group  $G$  acts isometrically on  $\mathcal{B}_E^1$  via the left regular representation. In particular, the space  $[C_u(G, E)]_\infty$  is a  $G$ -bimodule.*
- (iv) *For every  $u \in \mathcal{B}_\mathbb{C}^{\mathbf{m}}$  and  $a \in \mathcal{B}_E^{\mathbf{m}'}$ , their product,  $ua$ , belongs to  $\mathcal{B}_E^{\mathbf{m}\mathbf{m}'}$ . Moreover, the associated bilinear map:*

$$\mathcal{B}_\mathbb{C}^{\mathbf{m}} \times \mathcal{B}_E^{\mathbf{m}'} \rightarrow \mathcal{B}_E^{\mathbf{m}\mathbf{m}'}$$

*is continuous.*



(v) For every  $P \in \mathcal{U}(\mathfrak{g})$  and  $a \in \mathcal{B}_E^{\mathbf{m}}$ , the element  $\tilde{P}.a$  belongs to  $\mathcal{B}_E^{\mathbf{m}}$  and the map

$$\mathcal{B}_E^{\mathbf{m}} \rightarrow \mathcal{B}_E^{\mathbf{m}} : a \mapsto \tilde{P}.a$$

is continuous.

(vi) Assume the inverse,  $\frac{1}{\mathbf{m}}$ , of the weight  $\mathbf{m}$  vanishes at infinity. Then, the closure of  $\mathcal{D}_E$  in  $\mathcal{B}_E^{\mathbf{m}}$  contains  $\mathcal{B}_E^{\mathbf{m}'}$  for all  $\mathbf{m}'$  such that  $|\mathbf{m}'| < |\mathbf{m}|$ .

(vii) Let  $\mathbf{m} \in \mathcal{B}_{\mathbb{C}}^{\mathbf{m}}$  be a weight and consider any nowhere vanishing function  $\mathbf{m}_0$ . Then for all  $A \in \mathcal{B}_E^{\mathbf{m}_0}$  and all  $P \in \mathcal{U}(\mathfrak{g})$ , one has  $\tilde{P}.\left(\frac{A}{\mathbf{m}}\right) = \frac{1}{\mathbf{m}} A'$  where  $A'$  belongs to  $\mathcal{B}_E^{\mathbf{m}_0}$ .

*Proof.* The space  $C_u(G, E)$  is a Fréchet space for the seminorms  $\{|\cdot|_j^\infty\}$  defined as

$$|a|_j^\infty := \sup_G |a|_j.$$

Indeed,  $G$  being locally compact and countable at infinity the space  $C_b(G, E)$  is Fréchet (by the same argument as in the proof of Prop. 44.1 and Cor. 1. of [Tr67]). The subspace  $C_u(G, E)$  is then closed as a uniform limit of uniformly continuous functions is uniformly continuous (as it is seen by a 3-epsilon argument).

Moreover, the natural Fréchet topology on  $[C_u(G, E)]_\infty$  is induced by the set of seminorms  $\{|\cdot|_{P,j}\}$  on  $\mathcal{B}_E^1(G)$  defined as

$$|a|_{P,j} := \sup_G |\tilde{P}.F|_j.$$

Indeed, an element  $a \in [C_u(G, E)]_\infty$  is such that the function  $g \mapsto R_g^*a$  is smooth as a  $C_u(G, E)$ -valued function on  $G$ . In particular, for every  $P \in \mathcal{U}(\mathfrak{g})$ ,  $\tilde{P}.a$  is bounded and smooth. Reciprocally,  $G$  acts on  $\mathcal{B}_E^1$  via the right regular representation. Indeed, for all  $g \in G$ ,  $\tilde{P}_x(R_g^*a) = (\text{Ad}(g^{-1})P)^\sim|_{xg}.a$ . Hence  $\sup_x |\tilde{P}_x(R_g^*a)|_j = \sup_x |(\text{Ad}(g^{-1})P)^\sim|_{xg}.a|_j = \sup_x |(\text{Ad}(g^{-1})P)^\sim|_x.a|_j$  which is bounded for  $a \in \mathcal{B}_E^1$ . Note that the group  $G$  acts on  $\mathcal{B}_E^1$  via the left regular representation as well. Indeed,  $\sup |\tilde{P}.(L_g^*a)|_j = \sup |L_g^*(\tilde{P}.a)|_j = \sup |\tilde{P}.a|_j$ . The left regular representation action is in particular isometric. Note also that one has the inclusion:  $\mathcal{B}_E^1 \subset C_u(G, E)$ . Indeed, for  $a \in \mathcal{B}_{\mathbb{C}}^1$  the function  $\tilde{d}a : G \rightarrow \mathfrak{g}^* : x \mapsto da_x \circ L_{x\star_e}$  is such that  $\langle \tilde{d}a(x), H \rangle \leq c(H)$  where  $c : \mathfrak{g} \rightarrow \mathbb{C}$  is independent of  $x$ . One may moreover assume that the function  $c$  is continuous. Indeed, setting  $c_s(H) := \sup_x \{|\langle \tilde{d}a(x), H \rangle|\}$ , one observes that  $c_s(\lambda H) = |\lambda|c_s(H)$  and  $c_s(H + H') \leq c_s(H) + c_s(H')$ . Choosing a basis  $\{X_j\}$  of  $\mathfrak{g}$ , one then gets positive numbers  $\{m_j\}$  such that  $c_s(x^j X_j) \leq \sum_j m_j |x^j| =: c(x^j X_j)$ . Now, for fixed  $H$ , one observes that  $|a(x \exp(tH)) - a(x)| = |da_{x \exp(\tau H)}(\tilde{H})t| = |\langle \tilde{d}a(x \exp(\tau H)), H \rangle t|$  where  $\tau \in [0, t]$ . Therefore  $|a(x \exp(tH)) - a(x)| \leq c(H)|t|$ . Choosing a Euclidean scalar product on  $\mathfrak{g}$ , and denoting by  $B_r$  the open ball of radius  $r$  in  $\mathfrak{g}$ , one observes that for all  $x$  in  $G$ , one has  $|a(x \exp(B_r)) - a(x)| \leq \max_{|H|=1} (c) r$ ; hence the uniform continuity of  $a$ . To show that  $a \in \mathcal{B}_{\mathbb{C}}^1$  is a differentiable vector, we observe that

$$\begin{aligned} \sup_x \left\{ \left| \frac{1}{t} (a(x \exp(tX)) - a(x)) - \tilde{X}_x.a \right| \right\} &= \sup_x \{ |\tilde{X}_{x \exp(\tau X)}.a - \tilde{X}_x.a| \} \quad (\tau \in [0, t]) \\ &= |\tau| \sup_x \{ |\tilde{X}_{x \exp(\sigma X)}^2.a| \} \quad (\sigma \in [0, \tau]) \\ &\leq |t| \sup \{ |\tilde{X}^2.a| \}, \end{aligned}$$

which tends to zero together with  $t$ . This yields differentiability at the unit element. One gets it everywhere else by observing that

$$\tilde{X}.R^*(a) = R^*(\tilde{X}.a). \quad (11)$$

An induction on the order of derivation implies  $\mathcal{B}_{\mathbb{C}}^1 \subset [C_u(G)]_\infty$ , and the  $E$ -valued case is entirely similar. The assertion concerning the topology follows from the definition of the topology on smooth vectors [Wa72] and from (11) again.

Now, the notion of weight implies that  $a$  belongs to  $\mathcal{B}_E^{\mathbf{m}}$  iff  $\frac{1}{\mathbf{m}}a$  belongs to  $\mathcal{B}_E^1$ ; the non-constant weight case then follows.

Items (iv) and (vii) follows from Leibniz' rule while item (v) is obvious.

For the last assertion, we consider, similarly as in [Di78], a cut-off  $\varphi \in \mathcal{D}$  such that  $\varphi|_{B_1} = 1$  and  $\varphi|_{G \setminus B_2} = 0$ . Setting  $\varphi_n(x) := \varphi(\frac{1}{n}x)$ , we observe that  $\sup\{|\frac{1}{\mathbf{m}} \tilde{P} \cdot (1 - \varphi_n)|_j\}$  tends to zero when  $n$  tends to infinity for every  $P \in \mathcal{U}(\mathfrak{g})$ . Which amounts to say that  $\{\varphi_n\}$  converges to 1 in  $\mathcal{B}_C^{\mathbf{m}}$ . The latter combined with item (iv) yield item (vi).  $\blacksquare$

**Definition 3.2** Consider a function  $S \in C^\infty(G, \mathbb{R})$ . An element  $\mathbf{P}$  of  $\mathcal{U}(\mathfrak{g})$  is called  **$S$ -adapted** if the following conditions hold

(i) the function  $\mathbf{m}_\mathbf{P} := e^{-iS} \tilde{\mathbf{P}} \cdot e^{iS}$  is a weight;

(ii) its inverse  $\frac{1}{\mathbf{m}_\mathbf{P}}$  is integrable with respect to a left-invariant Haar measure on  $G$ .

If such an element  $\mathbf{P}$  exists then one calls  $S$  a **phase** on  $G$ . Moreover, for every weight  $\mathbf{m}_0$  such that  $\frac{\mathbf{m}_0}{\mathbf{m}_\mathbf{P}}$  is integrable, we call an **amplitude** (adapted to  $S$ ) any element of  $\mathcal{B}_E^{\mathbf{m}_0}$ .

**Remark 3.1** Note that the product of an amplitude by an element of  $\mathcal{B}_C^1$  is again an amplitude.

One then has

**Definition 3.3** Assume  $\mathbf{P} \in \mathcal{U}(\mathfrak{g})$  is  $S$ -adapted and self-adjoint<sup>3</sup>. Consider any weight  $\mathbf{m}_0$  such that  $\frac{\mathbf{m}_0}{\mathbf{m}_\mathbf{P}}$  is integrable. For  $A \in \mathcal{D}_E$ , an integration<sup>4</sup> by parts yields  $\int_G e^{iS} A = \int_G e^{iS} \tilde{\mathbf{P}} \cdot (\frac{A}{\mathbf{m}_\mathbf{P}})$ . Moreover, Lemma 3.1 implies that the linear map  $\mathcal{D}_E \rightarrow E : A \mapsto \int e^{iS} A$  extends by continuity to a continuous linear map:

$$\int e^{iS} : \mathcal{B}_E^{\mathbf{m}_0} \rightarrow E. \quad (12)$$

The latter is called the **oscillatory integral** on  $\mathcal{B}_E^{\mathbf{m}_0}$ .

## 3.2 Hyperbolic Laplacian and the canonical phase

**Lemma 3.2** On the affine group  $\mathbb{S}$ , one has

$${}^\tau \tilde{E} = -\tilde{E}; {}^\tau \tilde{H} = -\tilde{H} + 2;$$

and the operator

$$\tilde{B} := \alpha \tilde{H}^2 + \beta \tilde{E}^2 + \gamma \frac{1}{2} (\tilde{E} \tilde{H} + \tilde{H} \tilde{E}) - 2\alpha \tilde{H} - \gamma \tilde{E}$$

is self-adjoint for every data of  $\alpha, \beta, \gamma$  in  $\mathbb{R}$ .

In particular, the Laplace operator associated with any Damek-Ricci Riemannian structure on  $\mathbb{S}$  ([DR92]) is of the above form. The analysis which follows may be performed with any of the latter, however, for simplicity we shall only consider the hyperbolic Laplacian:

$$\Delta := 2(\tilde{H}^2 + \tilde{E}^2 - 2\tilde{H}).$$

**Proposition 3.1** The canonical two-point function on  $\mathbb{S} = M$ :

$$S_{\text{can}} := \sinh(2a_1)\ell_2 - \sinh(2a_2)\ell_1$$

is a phase on  $\mathbb{S} \times \mathbb{S}$ .

Moreover, every weight  $\mathbf{m}_0 = \mathbf{m}_0(a_1, a_2)$  such that

$$\frac{\mathbf{m}_0}{\sqrt{\cosh(2(a_1 - a_2)) \cosh(4(a_1 + a_2)) \cosh(4(a_1 - a_2))}} \in L^1_{(a_1, a_2)}$$

is an adapted amplitude.

<sup>3</sup>The latter condition of self-adjointness stands there for simplicity, but it is not essential.

<sup>4</sup>For standard definitions and properties about  $L^1(G, E)$ , we refer to [Gr52].

*Proof.* Define, for  $\phi \in C^\infty(\mathbb{S} \times \mathbb{S})$ ,

$$\Delta \phi := (\Delta \otimes 1 + 1 \otimes \Delta) \phi.$$

A computation shows that

$$\Delta^2 e^{iS_{\text{can}}} = (\mathcal{Q}_4 + \mathcal{Q}_2 + \mathbf{c} + i\mathcal{Q}_3) e^{iS_{\text{can}}}.$$

where

$$\mathcal{Q}_4 := 128 \left( \begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix} \begin{pmatrix} \cosh[4a_2] & \sinh[2(a_1 + a_2)] \\ \sinh[2(a_1 + a_2)] & \cosh[4a_1] \end{pmatrix} \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} \right)^2;$$

$$\begin{aligned} \mathcal{Q}_2 := & \ell_2^2 (528 - 2752 \cosh(4a_1) + 16 \cosh(8a_1) + 16 \cosh(4(a_1 - a_2)) + \\ & 16 \cosh(4(a_1 + a_2)) + 896 \sinh(4a_1)) + \ell_1^2 (528 + 16 \cosh(4(a_1 - a_2)) - \\ & 2752 \cosh(4a_2) + 16 \cosh(8a_2) + 16 \cosh(4(a_1 + a_2)) + 896 \sinh(4a_2)) + \\ & \ell_1 \ell_2 (-1024 \cosh(2(a_1 - a_2)) + 3328 \cosh(2(a_1 + a_2)) - \\ & 3968 \sinh(2(a_1 + a_2)) + 32 \sinh(2(3a_1 + a_2)) + \\ & 32 \sinh(2(a_1 + 3a_2))) ; \end{aligned}$$

$$\begin{aligned} \mathcal{Q}_3 := & \ell_2^3 ((128) \cosh(2a_1) + (128) \cosh(6a_1) + \\ & (384) \sinh(2a_1) - (896) \sinh(6a_1)) \\ & + \ell_1^3 ((-128) \cosh(2a_2) - \\ & (128) \cosh(6a_2) - (384) \sinh(2a_2) + (896) \sinh(6a_2)) + \\ & \ell_1 ((16) \cosh(4a_1 - 2a_2) + (880) \cosh(2a_2) - \\ & (16) \cosh(6a_2) - (112) \cosh(2(2a_1 + a_2)) - \\ & (48) \sinh(4a_1 - 2a_2) - (1552) \sinh(2a_2) + \\ & (144) \sinh(6a_2) + (48) \sinh(2(2a_1 + a_2))) + \\ & \ell_1 \ell_2^2 ((-128) \cosh(4a_1 - 2a_2) + (768) \cosh(2a_2) - \\ & (1920) \cosh(2(2a_1 + a_2)) - (384) \sinh(4a_1 - 2a_2) + \\ & (256) \sinh(2a_2) + (1152) \sinh(2(2a_1 + a_2))) + \\ & \ell_2 ((-880) \cosh(2a_1) + (16) \cosh(6a_1) - \\ & (16) \cosh(2(a_1 - 2a_2)) + (112) \cosh(2(a_1 + 2a_2)) + \\ & (1552) \sinh(2a_1) - (144) \sinh(6a_1) - (48) \sinh(2(a_1 - 2a_2)) - \\ & (48) \sinh(2(a_1 + 2a_2))) + \ell_1^2 \ell_2 ((-768) \cosh(2a_1) + \\ & (128) \cosh(2(a_1 - 2a_2)) + (1920) \cosh(2(a_1 + 2a_2)) - \\ & (256) \sinh(2a_1) - (384) \sinh(2(a_1 - 2a_2)) - \\ & (1152) \sinh(2(a_1 + 2a_2))) ; \end{aligned}$$

and

$$\begin{aligned} \mathbf{c} := & 202 - 232 \cosh(4a_1) + \cosh(8a_1) + 2 \cosh(4(a_1 - a_2)) - 232 \cosh(4a_2) + \\ & \cosh(8a_2) + 2 \cosh(4(a_1 + a_2)) + 64 \sinh(4a_1) + 64 \sinh(4a_2). \end{aligned}$$

Let  $C$  be a positive constant and set  $\mathbf{m}_C := e^{-iS} (C + \Delta^2) e^{iS}$ . On then has  $|\mathbf{m}_C| \geq |\mathcal{Q}_4 + \mathcal{Q}_2 + \mathbf{c} + C|$ . Let  $\Xi = \Xi(a_1, a_2)$  be a continuous function. The integral  $\int_{\mathbb{S} \times \mathbb{S}} \frac{\Xi}{\mathbf{m}_C}$  exists provided the following integral

$$\mathcal{I} := \int_{\mathbb{S} \times \mathbb{S}} \frac{\Xi}{\mathcal{Q}_4 + \mathcal{Q}_2 + \mathbf{c} + C}$$

does.

In order to establish the existence of the above integral  $\mathcal{I}$ , we first observe that the quadratic form  $A := \begin{pmatrix} \cosh[4a_2] & \sinh[2(a_1 + a_2)] \\ \sinh[2(a_1 + a_2)] & \cosh[4a_1] \end{pmatrix}$  is positive definite for all  $(a_1, a_2) =: a$ . Indeed its determinant equals  $[\cosh(2(a_2 - a_1))]^2$ .

We then analyse the behaviour of the quadratic form  $B$  defined as  $\mathcal{Q}_2 =: {}^\tau \ell B \ell$  with  $\ell := (\ell_1, \ell_2)$ . One observes that its trace is strictly positive for large values of  $|a|$  as its dominant term is greater than  $\cosh(8a_1) + \cosh(a_2)$ . Also for large  $|a|$ , its determinant behaves as  $128(\cosh(12a_1 - 4a_2) + \cosh(4(a_1 - 3a_2)) + 2 \cosh(8(a_1 - a_2))) = 128 \times 4(\cosh(8(a_1 - a_2)) \cosh^2(2(a_1 + a_2)))$ . Therefore  $B$  is positive definite for large values of  $|a|$ , say for  $|a| > \mathbf{a}_0 > 0$ .

Note that for large values of  $|a|$ , say  $|a| > \mathbf{a}_0$  as well, the independent term  $\mathbf{c}$  behaves (at least) as  $\cosh(8a_1) + \cosh(8a_2) = 2 \cosh(4(a_1 + a_2)) \cosh(4(a_1 - a_2))$ .

Now, one has

$$\mathcal{I} = \int \frac{\Xi}{({}^\tau \ell A \ell)^2 + {}^\tau \ell B \ell + \mathbf{c} + C},$$

which equals:

$$\int \frac{\Xi}{({}^\tau \ell \Lambda \ell)^2 + {}^\tau \ell U B {}^\tau U \ell + \mathbf{c} + C},$$

after changing the variables following  $\ell \rightarrow U \ell$  where  $U \in SO(2)$  is such that  ${}^\tau U \Lambda U = A$  with  $\Lambda$  diagonal (and positive definite). Hence setting  $x := \Lambda^{\frac{1}{2}} \ell$ , one gets:

$$\mathcal{I} = \int \frac{\Xi}{\sqrt{\det(A)} (x^4 + {}^\tau x \Lambda^{-\frac{1}{2}} U B {}^\tau U \Lambda^{-\frac{1}{2}} x + \mathbf{c} + C)} dx da.$$

Passing in polar coordinates  $x =: r e^{i\theta}$ , and setting

$$\beta := {}^\tau e^{i\theta} \Lambda^{-\frac{1}{2}} U B {}^\tau U \Lambda^{-\frac{1}{2}} e^{i\theta},$$

one obtains

$$\mathcal{I} = \int_{\mathbb{R}^2} \frac{\Xi}{\sqrt{\det(A)}} \int_0^{2\pi} \int_0^\infty \frac{r}{r^4 + \beta r^2 + \mathbf{c} + C} dr d\theta da,$$

or, after  $r \rightarrow r^2$ ,

$$\mathcal{I} = \frac{1}{2} \int_{\mathbb{R}^2} \frac{\Xi}{\sqrt{\det(A)}} \int_0^{2\pi} \int_0^\infty \frac{1}{r^2 + \beta r + \mathbf{c} + C} dr d\theta da.$$

Now, for  $|a| \geq \mathbf{a}_0$ ,  $\beta$  is strictly positive. Hence  $r^2 + \beta r + \mathbf{c} + C > r^2 + \mathbf{c} \sim r^2 + 2 \cosh(4(a_1 + a_2)) \cosh(4(a_1 - a_2))$ . Therefore the quantity

$$\mathcal{I}_+ := \frac{1}{2} \int_{|a| > \mathbf{a}_0} \frac{\Xi}{\sqrt{\det(A)}} \int_0^{2\pi} \int_0^\infty \frac{1}{r^2 + \beta r + \mathbf{c} + C} dr d\theta da,$$

is bounded by

$$\gamma \int_{|a| > \mathbf{a}_0} \frac{\Xi}{\sqrt{2 \det(A) \cosh(4(a_1 + a_2)) \cosh(4(a_1 - a_2))}} da \int_0^\infty \frac{1}{s^2 + 1} ds,$$

where  $\gamma$  is a positive constant.

Regarding the quantity:

$$\mathcal{I}_- := \frac{1}{2} \int_{|a| \leq \mathbf{a}_0} \frac{\Xi}{\sqrt{\det(A)}} \int_0^{2\pi} \int_0^\infty \frac{1}{r^2 + \beta r + \mathbf{c} + C} dr d\theta da,$$

one may adapt the constant  $C$  in such a way that  $\beta r + \mathbf{c} + C \geq 1$  for all  $(r, \theta, a)$  in the compact set  $[0, 1] \times [0, 2\pi] \times \{|a| \leq \mathbf{a}_0\}$ . Which yields a finite quantity  $\mathcal{I}_-$  for any data of a continuous function  $\Xi$ .  $\blacksquare$

### 3.3 Deformed products

We denote by  $\mathcal{D}_E$ ,  $\mathcal{S}_E$  and  $\mathcal{S}'_E$  respectively the space of  $E$ -valued smooth compactly supported functions on  $\mathbb{S}$ , the space of  $E$ -valued smooth rapidly decreasing functions on  $\mathbb{S}$  and the space of  $E$ -valued tempered distributions on  $\mathbb{S}$ . By nuclearity and denoting by  $\hat{\otimes}_\pi$  the projective tensor product, one has the following isomorphisms [Tr67]:  $\mathcal{D}_E \simeq \mathcal{D} \hat{\otimes}_\pi E$ ,  $\mathcal{S}_E \simeq \mathcal{S} \hat{\otimes}_\pi E$  and  $\mathcal{S}'_E \simeq \mathcal{S}' \hat{\otimes}_\pi E$ . In particular, the partial Fourier transform  $\mathcal{F}$  induces the following automorphisms:  $\mathcal{F} \hat{\otimes}_\pi \text{Id}_E : \mathcal{S}_E \rightarrow \mathcal{S}_E$  and  $\mathcal{F} \hat{\otimes}_\pi \text{Id}_E : \mathcal{S}'_E \rightarrow \mathcal{S}'_E$ .

**Lemma 3.3** *The linear map:*

$$\mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}'(\mathbb{R}) : \varphi \mapsto \varphi \circ \text{arcsinh}$$

*is continuous w.r.t. the strong topology on  $\mathcal{S}'(\mathbb{R})$ .*

*Proof.* First observe that the linear map

$$\mathcal{S} \rightarrow \mathcal{S} : \varphi \mapsto [\tilde{\varphi} : t \mapsto \cosh(t)\varphi(\sinh(t))]$$

is bounded. Now, assume  $\{\phi_n\}$  is a sequence of Schwartz functions that tends to zero in the  $\mathcal{S}$ -topology and consider the corresponding sequence  $\{T_n := \phi_n \circ \text{arcsinh}\}$  in  $\mathcal{S}'(\mathbb{R})$ . One then has  $\langle T_n, \varphi \rangle = \int \phi_n \tilde{\varphi} = \langle \phi_n, \tilde{\varphi} \rangle$ , which tends to zero uniformly for  $\varphi$  running in any bounded subset of  $\mathcal{S}(\mathbb{R})$  as the natural inclusion  $\mathcal{S}(\mathbb{R})$  in its strong dual  $\mathcal{S}'(\mathbb{R})$  is continuous.  $\blacksquare$

By nuclearity and for every Schwartz multiplier  $\Theta \in \mathcal{O}_M(\mathbb{R})$  ([Tr67] p. 275), one gets a continuous linear injection:

$$U^\Theta \hat{\otimes}_\pi \text{Id}_E : \mathcal{S}_E \longrightarrow \mathcal{S}'_E.$$

We then set

$$\mathcal{E}_E^\Theta := U^\Theta \hat{\otimes}_\pi \text{Id}_E(\mathcal{S}_E).$$

At last, we denote by  $\square$  the differential operator on  $C^\infty(\mathbb{S} \times \mathbb{S})$  defined as  $\square F := (C + \Delta^2) \left( \frac{1}{m_C} F \right)$ . We observe

**Lemma 3.4** (i) *Let  $\Theta \in C^\infty(\mathbb{R})$  be nowhere vanishing and satisfying the property that there exists  $N \in \mathbb{R}$  such that one may find  $C > 0$  with  $|\frac{d^r}{dt^r} \Theta(t)| \leq C(1 + |t|)^{N-r}$ . Then the function (see Theorem 2.1)*

$$\Theta(x_1, x_2) := \frac{\Xi(a_1) \Xi(-a_2)}{\Xi(a_1 - a_2)}$$

*belongs to  $\mathcal{B}_\mathbb{C}^1(\mathbb{S} \times \mathbb{S})$ .*

(ii) *The function*

$$\mathbf{A}_{\text{can}}(x_1, x_2) := \sqrt{|\text{Jac}_\Phi(e, x_1, x_2)|}$$

*is an amplitude on  $\mathbb{S} \times \mathbb{S}$  adapted to  $S_{\text{can}}$ .*

We now consider a strongly continuous isometric action  $\alpha$  of  $\mathbb{S}$  on a Fréchet algebra  $(\mathbb{A}, \mu_\mathbb{A})$  and denote by  $\mathbb{A}_\infty$  the associated space of smooth vectors.

**Corollary 3.1** *Let  $\Theta \in C^\infty(\mathbb{R})$  be as in Lemma 3.4. For all  $a, b \in \mathbb{A}_\infty$ , the following oscillatory integral*

$$a \star_\theta^\Theta b := \frac{1}{\theta^2} \int e^{\frac{i}{\theta} S_{\text{can}}} \Theta \mathbf{A}_{\text{can}} \mu_\mathbb{A}(\alpha(a) \otimes \alpha(b)) \quad (13)$$

*is well defined as an element of  $\mathbb{A}_\infty$ .*

**Theorem 3.1** *The pair  $(\mathbb{A}_\infty, \star_\theta^\Theta)$  is an associative topological algebra.*

*Proof.* The space  $\mathcal{E}_\theta(\mathbb{S}) \hat{\otimes} \mathbb{A}_\infty$  naturally inherits a structure of associative algebra denoted by  $\hat{\star}$  from the one on  $\mathcal{E}_\theta(\mathbb{S})$ . Consider  $a, b, c \in \mathbb{A}_\infty$ , then we have

$$\begin{aligned}
(a \star b) \star c &= \int \tilde{K}(e, g_1, g_2) \left( \alpha_{g_1} \int \tilde{K}(e, h_1, h_2) \alpha_{h_1}(a) \alpha_{h_2}(b) dh_1 dh_2 \right) \alpha_{g_2}(c) dg_1 dg_2 \\
&= \int e^{iS_e(g_1, g_2)} \square_{(g_1, g_2)} \left( \alpha_{g_1} \left( \int \tilde{K}(e, h_1, h_2) \alpha_{h_1}(a) \alpha_{h_2}(b) dh_1 dh_2 \right) \alpha_{g_2}(c) \right) dg_1 dg_2 \\
&= \int e^{iS_e(g_1, g_2)} \square_{(g_1, g_2)} \left( \alpha_{g_1} \left( \int e^{iS_e(h_1, h_2)} \square_{(h_1, h_2)} (\alpha_{h_1}(a) \alpha_{h_2}(b)) dh_1 dh_2 \right) \alpha_{g_2}(c) \right) dg_1 dg_2 \\
&= \int \int e^{i(S_e(g_1, g_2) + S_e(h_1, h_2))} \square_{(g_1, g_2)} \square_{(h_1, h_2)} (\alpha_{g_1}(\alpha_{h_1}(a) \alpha_{h_2}(b))) \alpha_{g_2}(c) dh_1 dh_2 dg_1 dg_2 \\
&= \int \int e^{i(S_e(g_1, g_2) + S_e(h_1, h_2))} \square_{(h_1, h_2)} \square_{(g_1, g_2)} (\alpha_{g_1 h_1}(a) \alpha_{g_1 h_2}(b)) \alpha_{g_2}(c) dh_1 dh_2 dg_1 dg_2 \\
&= \int \int e^{i(S_e(g_1, g_2) + S(g_1, h_1, h_2))} \square_{(g_1^{-1} h_1, g_1^{-1} h_2)} \square_{(g_1, g_2)} (\alpha_{h_1}(a) \alpha_{h_2}(b) \alpha_{g_2}(c)) dh_1 dh_2 dg_1 dg_2 \\
&= \lim_n \left[ \int \int e^{i(S_e(g_1, g_2) + S(g_1, h_1, h_2))} \square_{(g_1^{-1} h_1, g_1^{-1} h_2)} \square_{(g_1, g_2)} ([\alpha(a)]_n(h_1) [\alpha(b)]_n(h_2) [\alpha(c)]_n(g_2)) \right] \\
&= \lim_n \left[ \int \int e^{i(S_e(g_1, g_2) + S_e(h_1, h_2))} \square_{(h_1, h_2)} \square_{(g_1, g_2)} ([\alpha(a)]_n(g_1 h_1) [\alpha(b)]_n(g_1 h_2) [\alpha(c)]_n(g_2)) \right] \\
&= \lim_n \left[ \int \int e^{i(S_e(g_1, g_2) + S_e(h_1, h_2))} \square_{(g_1, g_2)} \square_{(h_1, h_2)} ([\alpha(a)]_n(g_1 h_1) [\alpha(b)]_n(g_1 h_2) [\alpha(c)]_n(g_2)) \right] \\
&= \lim_n \left[ \int e^{iS_e(g_1, g_2)} \square_{(g_1, g_2)} \int \left( e^{iS_e(h_1, h_2)} \square_{(h_1, h_2)} ([\alpha(a)]_n(g_1 h_1) [\alpha(b)]_n(g_1 h_2)) \right) [\alpha(c)]_n(g_2) \right] \\
&= \lim_n \left[ \int e^{iS_e(g_1, g_2)} \square_{(g_1, g_2)} \int \left( e^{iS_e(g_1, h_1, h_2)} \square_{(h_1, h_2)} ([\alpha(a)]_n(h_1) [\alpha(b)]_n(h_2)) \right) [\alpha(c)]_n(g_2) \right] \\
&= \lim_n \left[ \int e^{iS_e(g_1, g_2)} \square_{(g_1, g_2)} ([\alpha(a)]_n \hat{\star} [\alpha(b)]_n)(g_1) [\alpha(c)]_n(g_2) \right] \\
&= \lim_n \left[ \int \tilde{K}(e, g_1, g_2) ([\alpha(a)]_n \hat{\star} [\alpha(b)]_n)(g_1) [\alpha(c)]_n(g_2) \right] \\
&= \lim_n (([\alpha(a)]_n \hat{\star} [\alpha(b)]_n) \hat{\star} [\alpha(c)]_n)(e) \\
&= \lim_n ([\alpha(a)]_n \hat{\star} ([\alpha(b)]_n \hat{\star} [\alpha(c)]_n))(e) \\
&= a \star (b \star c).
\end{aligned}$$

■

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